Hermitian Geometries in Projective Space

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ABSTRACT

Using a Hermitian form on a vector space over GF (l), we produce a geometry on the associated projective space and prove that this geometry is characterized by its plane sections.

INTRODUCTION

From a vector space V of dimension $n+1$ over the Galois field GF (l), the classical projective space $P(n,l)$ is produced by taking the one and two dimensional subspaces of V to be the points and lines of $P(n,l)$. If there is a Hermitian form defined on V, then $l=k^2$ and we produce a *Hermitian geometry* on $P(n,l)$ by deleting the projective points which are images of vectors on which the form vanishes. For $n=2$, the resulting *Hermitian planes* are: *affine planes* (points on one projective line deleted), *ultraaffine planes* (points on $k+1$ projective lines through a fixed point deleted), and *planes with semiovals* (k^3+1 projective points deleted, no $k+2$ of which are collinear). In this paper we characterize Hermitian geometries by their plane sections. Our main theorem states: if $\mathcal{G}$ is a geometry in $P(n,l)$ such that the planes of $\mathcal{G}$ are Hermitian and the points of $\mathcal{G}$ lie in no proper subspace of $P(n,l)$, then $\mathcal{G}$ is a Hermitian geometry.

1. DEFINITIONS AND RELATED WORK

In general, a *geometry* $\mathcal{G}$ is a finite collection of points and point sets called lines such that two points lie on at most one line. A *subgeometry* $\mathcal{S}$ of $\mathcal{G}$ is a subset of points of $\mathcal{G}$ together with all the lines through at least two points of the subset. The points in the *subgeometry generated by a subset* S of points of $\mathcal{G}$ are the points of S, all the points on lines joining points of S

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(the "first generation" points), all the points on lines joining first generation points to points in S and to other first generation points ("second generation"), and so on. A *plane of* $\mathcal{G}$ is the subgeometry generated by three noncollinear points. We say that a geometry $\mathcal{G}$ *is carried by* $P(n, l)$ if the points and lines of $\mathcal{G}$ are projective points and subsets of projective lines such that the points of $\mathcal{G}$ lie in no proper subspace of $P(n, l)$, *i.e.*, they span $P(n, l)$. Furthermore, we call a geometry *locally Hermitian* if each of its planes is a Hermitian plane. Thus, we may restate the main theorem as follows: a locally Hermitian geometry carried by $P(n, l)$ is a Hermitian geometry.

Theorems characterizing the classical projective and affine spaces by their plane sections serve as models for our work. A geometry in which every plane is a Desarguesian projective plane is a projective space $P(n, l)$ (see [2] for a proof). F. Buekenhout [2] and M. Hall [7] have obtained analogous results for affine spaces. Geometries with both projective and affine planes have been determined by J. Hall [6] and L. Teirlinck [13].

This paper completes a study of geometries in $P(n, l)$ produced by sesquilinear forms on the associated vector space V_{n+1}. We produce the *symplectic geometry* from an alternative bilinear form by deleting the projective lines which are images of the two dimensional subspaces of V_{n+1} on which the form is identically zero (totally isotropic lines) and the projective points whose preimages are in the radical of the form. The planes of a symplectic geometry are dual affine. M. Hale and the author [4] characterized such geometries as follows: a geometry with all planes dual affine and carried by $P(n, l)$ for $l > 2$ is a symplectic geometry. M. Hale [5] had determined locally dual affine geometries in $P(n, 2)$ earlier. If the form is symmetric, we produce the *orthogonal geometry* by deleting the projective points that are images of vectors on which the form is zero (isotropic points). The planes of an orthogonal geometry are: affine planes, hyperaffine planes (points on two projective lines deleted), punctured planes (one point deleted), and planes with ovals ($l+1$ points deleted, no three of which are collinear). The author [3] recently proved that a geometry carried by $P(n, l)$ for l odd such that every plane is orthogonal is an orthogonal geometry.

Much work has been done on the characterization of the isotropic points and totally isotropic lines produced by a sesquilinear form. The set of points in $P(n, l)$ isotropic under a nonsingular Hermitian form on V_{n+1} is the canonical nonsingular Hermitian variety. Each line in $P(n, l)$ meets this set in 1, $k+1$, or $l+1$ points. Tallini Scafati [12] began a characterization of these varieties using this property. This work has been expanded by J. W. P. Hirschfeld and J. A. Thas [8,9]. Many characterizations by line or plane sections of sets of isotropic points have involved only nonsingular forms. The geometries we consider contain a mix of planes produced by both singular and nonsingular forms.

In this paper we work over the Galois field GF(l) for $l=k^2$ and assume that ϕ is the involutory automorphism on GF(l) defined by $\alpha^\phi=\alpha^{k+1}$ for $\alpha\in$GF(l). We write P_n for $P(n,l)$ and denote the projective subspace spanned by points $m,n,\ldots$ and lines $M,N,\ldots$ by $\langle m,n,\ldots,M,N,\ldots\rangle$. If $p\in P_n$, we let p^* denote a vector in the associated vector space V_{n+1} which generates p, *i.e.*, $p=\{\alpha p^*|\alpha\in\mathrm{GF}(l)\}$. For a subspace T of P_n, T^* is the subspace of V spanned by $\{p^*|p\in T\}$.

In Sec. 2, we describe the Hermitian planes and derive properties of locally Hermitian geometries. In Sec. 3, we discuss our strategy for proving the main theorem and develop a notion of perpendicularity. Sections 4 and 5 contain the proofs of our theorem in the P_3 and the general case respectively.

2. LOCALLY HERMITIAN GEOMETRIES

We begin this section with a description of the Hermitian planes carried by a projective plane P. Let $\{m^*,n^*,p^*\}$ be a basis for the vector space V associated with P; the possible Hermitian forms on V are defined with respect to this basis. If the form is given by

$$\begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $b\neq 0$, then the points on $\langle n,p\rangle$ are deleted to produce the affine plane p'. If the form has the matrix

$$\begin{bmatrix} 0 & c & 0 \\ c^\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $c\neq 0$, then the points on $k+1$ lines through p are deleted to produce the ultraaffine plane P'. The resulting lines of P' contain l or $l-k$ points; we say these lines have *length* l or $l-k$. If the matrix for the form is

$$\begin{bmatrix} 0 & c & 0 \\ c^\phi & 0 & 0 \\ 0 & 0 & b \end{bmatrix}$$

for $b=-(c+c^\phi)\neq 0$, the Hermitian plane P' is produced by deleting k^3+1

points of P, no $k+2$ of which are collinear. The plane P', called a plane with a semioval, has lines of length l and $l-k$. For proof that this is a complete list of the possible Hermitian planes, see [11].

If $\mathcal{H}$ is the Hermitian geometry produced on P_n by a form on V_{n+1}, then every projective line or plane is deleted or carries a line or plane of $\mathcal{H}$. The following series of results shows that a locally Hermitian geometry $\mathcal{G}$ carried by P_n behaves the same way. If a projective line L carries a line L' of $\mathcal{G}$, we refer to the points on $L \backslash L'$ as *deleted points for* L'. Also a projective line or plane containing no $\mathcal{G}$ points is called *deleted*.

LEMMA 1. *If $x, y \in \mathcal{G}$, then $\langle x, y \rangle$ carries a line of $\mathcal{G}$.*

Proof. Since the points of $\mathcal{G}$ span P_n, there is a point z of $\mathcal{G}$ not on $\langle x, y \rangle$. These three points generate a plane P' of $\mathcal{G}$ carried by $\langle x, y, z \rangle$. By inspection of the possible configurations of P', we see that $\langle x, y \rangle$ carries a line in P'. ■

LEMMA 2. *If $\mathcal{G}$ is a locally Hermitian geometry carried by P_3, then every projective plane P carries a Hermitian plane of $\mathcal{G}$ or is deleted.*

Proof. Assuming that P is not deleted, we can find a point x of $\mathcal{G}$ in P. By the spanning property of $\mathcal{G}$, we can also find points y, z in $\mathcal{G}$ outside P; furthermore, by dimensionality, $\langle x, y, z \rangle$ meets P in a line L through x. Applying these arguments again, there is a $\mathcal{G}$ point $u \notin \langle x, y, z \rangle$ such that $\langle x, y, u \rangle$ meets P in a line M. We note that $L \neq M$; otherwise, $u \in \langle y, M \rangle = \langle y, L \rangle = \langle x, y, z \rangle$. Both $\langle x, y, z \rangle$ and $\langle x, y, u \rangle$ carry planes of $\mathcal{G}$, so L and M carry lines of $\mathcal{G}$, and thus P carries a plane of $\mathcal{G}$. ■

LEMMA 3. *Assume that $\mathcal{G}$ is a locally Hermitian geometry carried by P_n and that $\mathcal{S}$ is the subgeometry of $\mathcal{G}$ generated by four noncoplanar points p, q, r, and s of $\mathcal{G}$. Then $\mathcal{S}$ is a locally Hermitian geometry carried by $P_3 = \langle p, q, r, s \rangle$.*

Proof. We first claim that the points of $\mathcal{G}$ in P_3 are the points of $\mathcal{S}$. If x is a $\mathcal{G}$ point on $\langle p, q \rangle$, then clearly $x \in \mathcal{S}$. For $x \notin \langle p, q \rangle$, the plane $\langle x, p, q \rangle$ carries a Hermitian plane of $\mathcal{G}$ and meets $\langle r, s, p \rangle$ in a line L through p. The $\mathcal{G}$ line carried by L is a line of $\mathcal{S}$, since $L \subseteq \langle r, x, p \rangle$, and it follows that the $\mathcal{G}$ lines in $\langle L, q \rangle$ are lines of $\mathcal{S}$. Since x lies on one of these lines, $x \in \mathcal{S}$. We now prove that $\mathcal{S}$ is locally Hermitian by arguing that every plane P of P_3 containing at least one $\mathcal{G}$ point x has three noncollinear $\mathcal{G}$ points and thus carries a plane of $\mathcal{G}$.

Case 1. $\langle p, q\rangle$ *or* $\langle r, s\rangle$ *is in* P. We may assume that $\langle p, q\rangle$ lies in P. Then $\langle q, r, s\rangle$ meets P in a line $L \neq \langle p, q\rangle$, and so P contains three noncollinear points of $\mathcal{G}$.

Case 2. Neither $\langle p, q\rangle$ *nor* $\langle r, s\rangle$ *is in* P. Without loss of generality, we may assume $p, q, r, s \notin P$. Since $\langle p, q\rangle$ and $\langle r, s\rangle$ do not intersect, we may also assume that x does not lie on $\langle p, q\rangle$. The planes $\langle x, p, q\rangle$ and $\langle q, r, s\rangle$ meet P in lines L and M respectively. We observe that $L \neq M$; otherwise, $p, q, r, s \in \langle q, L\rangle$. It follows that P contains three noncollinear $\mathcal{G}$ points.

That $\mathcal{S}$ is carried by P_3 follows from the fact that the two nonintersecting lines $\langle r, s\rangle$ and $\langle p, q\rangle$ generate a three dimensional projective space. ∎

Lemma 4. *Let G be a locally Hermitian geometry carried by P_n. Then*

(i) *every point of P_n lies on a projective line carrying a line of $\mathcal{G}$.*
(ii) *every line of P_n is deleted or carries a line of $\mathcal{G}$, and*
(iii) *every plane of P_n is deleted or carries a plane of $\mathcal{G}$.*

Proof. To prove (i), let $z \in P_n$.

Case 1. $z \in \langle x, y\rangle$ *such that* $x, y \in G$. By Lemma 1, $\langle x, y\rangle = \langle x, z\rangle$ carries a $\mathcal{G}$ line.

Case 2. $z \in \langle x, y\rangle$ *such that* $x \in \langle r, s\rangle$, $y \in \langle p, q\rangle$ *for* $r, s, p, q \in \mathcal{G}$. By Lemma 3, $<r, s, p, q>$ carries a locally Hermitian subgeometry of $\mathcal{G}$, and by Lemma 2, $\langle x, q, y\rangle$ carries a Hermitian plane of $\mathcal{G}$. Thus $\langle q, z\rangle$ carries a line of $\mathcal{G}$.

Case 3. $z \in <x, y>$ *such that* $x \in \mathcal{G}$, *and* $y \in <p, q>$ *for* $p, q \in \mathcal{G}$. Since $\mathcal{G}$ spans P_n, there is a point r in $\mathcal{G}$ outside of $\langle p, q, x\rangle$. Thus applying Lemmas 2 and 3 to $\langle r, x, p, q\rangle$, we have that $\langle q, z\rangle$ carries a line of $\mathcal{G}$.

Since the projective space generated by the points of $\mathcal{G}$ is P_n, all projective points fall into the following classes: points of $\mathcal{G}$, points on projective lines through two $\mathcal{G}$ points ("first generation"), points on lines through first generation and $\mathcal{G}$ points ("second generation"), and so on. The three cases show that no new points appear in the second generation. Thus (i) holds.

For (ii), let L be a projective line with at least one point x of G, and let $u \neq x$ be another point on L. By (i), there exists w in $\mathcal{G}$ such that $\langle u, w\rangle$ carries a $\mathcal{G}$ line and such that $\langle x, u, w\rangle$ carries a line or plane of $\mathcal{G}$. In either case L carries a $\mathcal{G}$ line.

To show that (iii) is true, we let P be a projective plane containing a point x of $\mathcal{G}$ and choose two lines L and M through x in P. By (ii), L and M carry $\mathcal{G}$ lines, and so P contains three noncollinear points of $\mathcal{G}$. ∎

3. STRATEGY AND PERPENDICULARITY

Our general strategy for proving that a locally Hermitian geometry $\mathcal{G}$ carried by P_n is Hermitian is similar to what we used in the symplectic and orthogonal cases (see [4] and [3]). A projective point s is *singular for* $\mathcal{G}$ if $s \notin \mathcal{G}$. (We could have defined singularity in terms of lines of $\mathcal{G}$ as we did in the symplectic and orthogonal cases. It is immediate that the singular points for $\mathcal{G}$ are the points not on any lines of $\mathcal{G}$. In contrast to geometries produced by alternate bilinear forms, if the projective point s lies on a line L which is deleted or carries a $\mathcal{G}$ line missing s, then s is singular for $\mathcal{G}$.)

We will prove $\mathcal{G}$ is a Hermitian geometry by finding a Hermitian form $(\,,\,)$ on the vector space V_{n+1} associated with P_n such that a point s is singular for $\mathcal{G}$ exactly when $(s^*, s^*)=0$. We do this in two steps: in the construction step, we define a form on V_{n+1} such that the Hermitian geometry $\mathcal{H}$ produced by the form agrees with $\mathcal{G}$ on certain subspaces of P_n; in the uniqueness step, we show that the agreement of $\mathcal{G}$ and $\mathcal{H}$ on these subspaces forces agreement on all of P_n.

These subspaces of agreement arise from a notion of perpendicularity. If s is a singular point of $\mathcal{G}$, we define the set $s^{\perp}$ to be $\{p \in P_n | \langle x, p\rangle$ is a deleted point or line or carries a line of length l in $\mathcal{G}\}$. We see from the next result that $s^{\perp}$ behaves as it should.

LEMMA 5. *If s is singular for $\mathcal{G}$, then $s^{\perp}$ is a projective hyperplane of P_n.*

Proof. We first show that $s^{\perp}$ is projective line closed by choosing x and y in $s^{\perp}$ and arguing that for any $w \in \langle x, y\rangle$, $\langle x, w\rangle$ is deleted or carries a $\mathcal{G}$ line of length l. If x, y, and s are collinear, then our result is immediate. Otherwise, $\langle x, y, s\rangle$ is a plane containing at least one deleted line through s or two projective lines through s carrying l-length lines of $\mathcal{G}$. It follows that $\langle x, y, s\rangle$ is deleted or carries an affine or ultraaffine plane of $\mathcal{G}$ in which all the projective lines through s are deleted or carry $\mathcal{G}$ lines of length l. Hence $w \in s^{\perp}$.

It remains to show that $s^{\perp}$ meets every projective line. Assume that for some line L of P_n, $s^{\perp} \cap L = \varnothing$. Then for any point x on L, $\langle x, s\rangle$ carries a $\mathcal{G}$ line of length $l-k$, and so there are $(l+1)k+1=k^3+k+1$ singular points for $\mathcal{G}$ in $\langle s, L\rangle$. However, there is no Hermitian plane configuration with exactly this number of singular points. Thus, $s^{\perp} \cap L \neq \varnothing$. ■

The next group of results will be used to find a Hermitian form on V_{n+1} which produces $\mathcal{G}$ on P_n.

LEMMA 6. *Let $\mathcal{H}$ be the Hermitian geometry on P_n produced by the Hermitian form (,) on V_{n+1}. If s is a singular point for $\mathcal{H}$, the projective point p is in $s^{\perp}$ exactly when $s^*, p^*)=0$.*

Proof. We note that by construction, $(s^*, s^*)=0$. Any point on $\langle s, p\rangle$ different from s can be written $\langle \alpha s^* + p^*\rangle$ for $\alpha \in \mathrm{GF}(l)$. Let $a_1=(s^*, p^*)$ and $a_2=(p^*, p^*)$; then $(\alpha s^*+p^*, \alpha s^*+p^*)=\alpha a_1+(\alpha a_1)^{\phi}+a_2$. It is straightforward to see that $a_1=0$ implies that $\langle s, p\rangle$ is deleted or carries a line of length l in $\mathcal{H}$.

Now let us assume $a_1\neq 0$ and show that this contradicts the fact that $p\in s^{\perp}$. If $\langle s, p\rangle$ carries an l-length line of $\mathcal{H}$, then $a_2\neq 0$ and we can find $\alpha \in \mathrm{GF}(l)$ such that

$$\alpha a_1+(\alpha a_1)^{\phi}=-a_2.$$

However, this implies that $\langle s, p\rangle$ contains more than one singular point for $\mathcal{H}$. If $\langle s, p\rangle$ is deleted, then $a_2=0$. Choosing α such that $\alpha a_1+(\alpha a_1)^{\phi}\neq 0$ leads to the conclusion that $\langle s, p\rangle$ is not deleted; again, a contradiction. Thus, $a_1=0$. ■

LEMMA 7. *Assume a projective line L carries a $\mathcal{G}$ line L' of length $l-k$. The Hermitian form (,) on the vector space V_2 associated with L is defined by the matrix*

$$\begin{bmatrix} 0 & a \\ a^{\phi} & 0 \end{bmatrix}$$

for $a\neq 0$ such that $a+a^{\phi}=0$, with respect to some basis for V_2.

Proof. Let m, n, and s be three singular points for L', and choose m^* and n^* such that $s=\langle m^*+n^*\rangle$. It follows from the singularity of m, n, and s that the matrix for (,) with respect to $\{m^*, n^*\}$ is

$$\begin{bmatrix} 0 & a \\ a^{\phi} & 0 \end{bmatrix}$$

where $a+a^{\phi}=0$. ■

COROLLARY 8. *Let $\mathcal{G}$ and $\mathcal{H}$ be two locally Hermitian geometries carried by P_n. Assume that L carries the $(l-k)$-length lines L' of G and L'' of $\mathcal{H}$. If* *there are three points m, n, and s on L singular for both geometries, then $L' = L''$.*

REMARK. We say G and H *agree on* L.

Proof. Choosing the basis $\{m^*, n^*\}$ as in Lemma 7, and $a \neq 0 \in \mathrm{GF}(l)$ such that $a + a^{\phi} = 0$, we define the Hermitian form (,) on V_2 by

$$\begin{bmatrix} 0 & a \\ a^{\phi} & 0 \end{bmatrix}$$

with respect to $\{m^*, n^*\}$. From Lemma 7, both L' and L'' are produced by this form and hence $L' = L''$. ■

LEMMA 9. *Let P be a projective plane carrying a plane P with semioval of $\mathcal{G}$. Let $c \in GF(l)$ such that $c + c^{\phi} \neq 0$. Then the Hermitian form (,) on V_3 associated with P which produces P is defined by*

$$\begin{bmatrix} 0 & c & 0 \\ c^{\phi} & 0 & 0 \\ 0 & 0 & b \end{bmatrix}$$

for $b = -(c + c^{\phi})$, with respect to some basis for V_3.

Proof. This proof is found in [10, p. 61]. It is useful to describe how the basis is chosen. Let m, n, and q be three noncollinear singular points for $\mathcal{G}$ in P, and let p be the point in P in $m^{\perp} \cap n^{\perp}$. If we choose m^*, n^*, and p^* such that $q = \langle m^* + n^* + p^* \rangle$ and c in $\mathrm{GF}(l)$ such that $c + c^{\phi} \neq 0$, then the Hermitian form defined by the above matrix produces P'. ■

COROLLARY 10. *Let $\mathcal{G}$ and $\mathcal{H}$ be two locally Hermitian geometries carried by P_n. Assume the plane P carries planes P' and P'' with semiovals in $\mathcal{G}$ and $\mathcal{H}$, respectively. If there are four points m, n, p, and q in P such that*

(i) *m, n, and q are noncollinear points singular for $\mathcal{G}$ and $\mathcal{H}$, and*
(ii) *p is a point in $\mathcal{G}$ and $\mathcal{H}$ such that $p \in m^{\perp} \cap n^{\perp}$ in both geometries,*
then $P' = P''$.

REMARK. We say that $\mathcal{G}$ and H *agree on* P.

Proof. From Lemma 9, P' and P'' are produced by the same Hermitian form, and so the singular points for $\mathcal{G}$ in P are the singular points for $\mathcal{H}$ in P. ■

4. LOCALLY HERMITIAN GEOMETRIES IN P_3

As in the symplectic and orthogonal geometries, the result in the three dimensional case is the key for the main theorem.

THEOREM 1. *If $\mathcal{G}$ is a locally Hermitian geometry carried by P_3, then $\mathcal{G}$ is a Hermitian geometry.*

Proof. We first assume that $\mathcal{G}$ has at least one ultraaffine plane or plane with a semioval. Then we can choose points m and n singular for $\mathcal{G}$ such that $\langle m, n\rangle$ carries a $\mathcal{G}$ line of length $l-k$. By dimensionality, $m^{\perp} \cap n^{\perp}$ is a projective line L. We will choose two points p, q on L and define a Hermitian form (,) on the vector space V_4 associated with P_3 using the basis $\{m^*, n^*, p^*, q^*\}$. The Hermitian geometry $\mathcal{H}$ produced by (,) will agree with $\mathcal{G}$ on $\langle m, n, p\rangle$ and L; we will show that this forces agreement on all of P_3. Our choice of a basis and our definition of the form (,) depend on how many $\mathcal{G}$ points lie on L.

Case 1. L is a deleted line. Let $s \neq m, n$ be a singular point for $\mathcal{G}$ on $\langle m, n\rangle$, and let p, q be any two points on L. Choose m^*, n^* such that $s = \langle m^* + n^*\rangle$ and $c \neq 0 \in \mathrm{GF}(l)$ such that $c + c^{\phi} = 0$. Then the form (,) defined on V_4 by

$$\begin{bmatrix} 0 & c & 0 & 0 \\ c^{\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

via $\{m^*, n^*, p^*, q^*\}$ produces a Hermitian geometry $\mathcal{H}$ which agrees with $\mathcal{G}$ on $\langle m, n\rangle$ and L. Furthermore, applying Lemma 6, we have that $L = m^{\perp} \cap n^{\perp}$ in $\mathcal{H}$.

We now show that $\mathcal{G}$ and H agree on every plane $\langle m, n, y\rangle$ for $y \in L$. Since $\langle m, y\rangle$ and $\langle n, y\rangle$ are deleted for $\mathcal{G}$ and $\mathcal{H}$, $\langle m, n, y\rangle$ carries an ultraaffine plane in both geometries. The deleted lines for each geometry are those through y and the singular points on $\langle m, n\rangle$. Since $\mathcal{G}$ and $\mathcal{H}$ agree on $\langle m, n\rangle$, they agree on $\langle m, n, y\rangle$.

Case 2. L carries an l-length line of $\mathcal{G}$. Let P be a $\mathcal{G}$ point on L. Then the plane $P = \langle m, n, p\rangle$ carries a plane with semioval in $\mathcal{G}$. Let j be a singular point for $\mathcal{G}$ in $P \backslash \langle m, n\rangle$. If we choose m^*, n^*, and p^* such that $j = \langle m + n + p^*\rangle$ and c in $\mathrm{GF}(l)$ such that $c + c^{\phi} \neq 0$, then the form defined via $\{m^*, n^*, p^*, q^*\}$ by

$$\begin{bmatrix} 0 & c & 0 & 0 \\ c^\phi & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $b = -(c + c^\phi)$, produces a Hermitian geometry $\mathcal{H}$ which agrees with $\mathcal{G}$ on P by Corollary 10 and on L. Furthermore, Lemma 6 implies that $m^\perp \cap n^\perp = L$ and $q^\perp = P_3$ for $\mathcal{H}$ as in $\mathcal{G}$.

We complete the proof in this case by showing that $\mathcal{G}$ and $\mathcal{H}$ agree on all planes $\langle m, n, y\rangle$ for $y \in L\backslash\{p\}$. Since $\mathcal{G}$ and $\mathcal{H}$ agree on $\langle m, n\rangle$ and $q^\perp = P_3$, the two geometries agree on the ultraaffine plane carried by $\langle m, n, q\rangle$. For $y \in L\backslash\{p, q\}$, $\langle m, y\rangle$ and $\langle n, y\rangle$ carry l-length lines in $\mathcal{G}$ and $\mathcal{H}$. Since $\langle j, q\rangle$ is deleted for both geometries, $\langle j, q\rangle$ meets $\langle m, n, y\rangle$ in a point $t \notin \langle m, n\rangle$ which is singular for $\mathcal{G}$ and $\mathcal{H}$. Applying Corollary 10 to $\mathcal{G}$ and $\mathcal{H}$ using points m, n, t, and y, we have that $\mathcal{G}$ and $\mathcal{H}$ agree on $\langle m, n, y\rangle$.

Case 3. L carries a $\mathcal{G}$ line of length $l-k$. Let p and q be points on L such that $p \in \mathcal{G}$ and q is singular for $\mathcal{G}$. Since $\langle m, p\rangle$ and $\langle n, p\rangle$ carry l-length lines of $\mathcal{G}$, the plane $P = \langle m, n, p\rangle$ carries a plane with semioval in $\mathcal{G}$ and we can find a singular point j in $P\backslash\langle m, n\rangle$. We choose m^*, n^*, p^* such that $j = \langle m^* + n^* + p^*\rangle$ and $c \in \mathrm{GF}(l)$ such that $b = -(c + c^\phi) \neq 0$. Furthermore, let $s \neq q$ be a singular point for $\mathcal{G}$ on L, and choose q^* such that $s = \langle p^* + q^*\rangle$. We define a Hermitian form via $\{m^*, n^*, p^*, q^*\}$ by

$$\begin{bmatrix} 0 & c & 0 & 0 \\ c^\phi & 0 & 0 & 0 \\ 0 & 0 & b & a \\ 0 & 0 & a^\phi & 0 \end{bmatrix},$$

where a is determined as follows.

Claim. There exists α in $\mathrm{GF}(l)$ such that $\langle p^* + \alpha q^*\rangle$ is singular for $\mathcal{G}$ and $\alpha \neq \alpha^\phi$. The k singular points for $\mathcal{G}$ on $L\backslash\{q\}$ are $\langle p^* + \gamma q^*\rangle$ for the nonzero values γ in $\mathrm{GF}(l)$. There are $k-1$ values $\beta \neq 0$ in $\mathrm{GF}(l)$ such that $\beta = \beta^\phi$. Thus, for some α, $\langle p^* + \alpha q^*\rangle$ is singular and $\alpha \neq \alpha^\phi$.

We now solve for a such that $\mathcal{G}$ and $\mathcal{H}$ will agree on P and L. The singularity of s and $\langle p^* + \alpha q^*\rangle$ implies that $b + a + a^\phi = 0$ and $b + \alpha a^\phi + \alpha^\phi a = 0$. Since $\alpha - \alpha^\phi \neq 0$, we have that $a = (1-\alpha)b/(\alpha - \alpha^\phi)$. Applying Corollary 8 to points m, n, and s on L and Corollary 10 to m, n, j, and p in P, we conclude that $\mathcal{G}$ and $\mathcal{H}$ agree on L and P. Furthermore, from Lemma 6 we have that $\mathcal{H}$ agrees with $\mathcal{G}$ on $m^\perp = \langle m, L \rangle$, $n^\perp = \langle n, L \rangle$, $p^\perp = \langle p, m, n \rangle$, and $q^\perp = \langle q, m, n \rangle$.

We complete this case by showing that $\mathcal{G}$ and $\mathcal{H}$ agree on $\langle x, L \rangle$ for all $x \in \langle m, n \rangle \setminus \{m, n\}$. When x is singular for $\mathcal{G}$ and $\mathcal{H}$, $\langle x, L \rangle$ carries an ultraaffine plane in both geometries. Since $\mathcal{G}$ and $\mathcal{H}$ agree on the singular points on L, they agree on the lines deleted in $\langle x, L \rangle$ and hence on the untraaffine plane carried by $\langle x, L \rangle$. When x is in $\mathcal{G}$ and $\mathcal{H}$, $\langle x, q \rangle$ and $\langle x, p \rangle$ carry l-length lines; let y be the point on $\langle x, p \rangle$ singular for $\mathcal{G}$ and $\mathcal{H}$. Since $\mathcal{G}$ and $\mathcal{H}$ agree on the singularity of y, s, and q and on $x \in q^\perp \cap y^\perp$, we apply Corollary 10 to conclude that they agree on $\langle x, L \rangle$.

Finally, if $\mathcal{G}$ has no planes with semiovals or ultraaffine planes, then all of the planes of $\mathcal{G}$ are affine. From the work of Buekenhout [2] and M. Hall [7] we know that $\mathcal{G}$ is the geometry produced by the form on V_4 defined by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to some basis. ■

5. PROOF OF THE MAIN THEOREM

We prove that a Hermitian geometry is characterized by its planes.

THEOREM 2. *If $\mathcal{G}$ is a locally Hermitian geometry carried by P_n, then $\mathcal{G}$ is a Hermitian geometry.*

Proof. We prove this theorem by induction on n; the $n=3$ case was done in Theorem 1. Let us now assume that the result holds for locally Hermitian geometries carried by P_m for $m<n$ and that $\mathcal{G}$ is such a geometry carried by P_n.

Case 1. *$\mathcal{G}$ contains a plane P' with semioval carried by some projective plane P.* Let m and n be singular points for $\mathcal{G}$ in P such that $\langle m, n \rangle$ carries a $\mathcal{G}$ line of length $l-k$, and let z be the $\mathcal{G}$ point in P in $m^\perp \cap n^\perp$.

Claim. The points of $\mathcal{G}$ in $m^\perp$ generate a locally Hermitian subgeometry $\mathcal{S}$ of $\mathcal{G}$ carried by $m^\perp$ such that the lines of $\mathcal{G}$ in $m^\perp$ are the lines of $\mathcal{S}$.

Since the points of $\mathcal{G}$ in $m^\perp$ generate $\mathcal{S}$, it follows that the lines of $\mathcal{G}$ in $m^\perp$ are those of $\mathcal{S}$ and that the planes of $\mathcal{S}$ are Hermitian. Thus it remains to prove that $\mathcal{S}$ is carried by $m^\perp$. We do this by showing that any singular point x for $\mathcal{G}$ in $m^\perp$ lies on a line in $m^\perp$ carrying a $\mathcal{G}$ line: the line $\langle x, z\rangle$ is in $m^\perp$ and carries a $\mathcal{G}$ line of length l or $l-k$.

By our induction hypothesis, $\mathcal{S}$ is the Hermitian geometry produced by some form $(\ ,\ )_1$ on the vector space V_n associated with $m^\perp$. We will extend $(\ ,\ )_1$ to a form on the vector space V_{n+1} associated with P_n such that the Hermitian geometry $\mathcal{H}$ produced by this form agrees with $\mathcal{G}$ on $m^\perp$ and P. Then we will prove that $\mathcal{G}$ and $\mathcal{H}$ agree on all of P_n.

We choose a basis for V_{n+1} as follows. Let $s\neq m, n$ be a singular point for $\mathcal{G}$ on $\langle m, n\rangle$, and choose m^*, n^* such that $s=\langle m^*+n^*\rangle$. Let $\{p_2^*,\dots,p_n^*\}$ be a basis for $(m^\perp\cap n^\perp)^*$; then $\{p_1^*=m^*, p_2^*,\dots,p_n^*\}$ and $\{n, p_2^*,\dots,p_n^*\}$ are bases for V_n and V_{n+1}, respectively. Now we define the form $(\ ,\ )$ on V_{n+1} with respect to $\{n^*, p_1^*,\dots,p_n^*\}$ by the matrix

$$\begin{bmatrix} 0 & c & 0 & \cdots & 0 \\ c^\phi & & & & \\ 0 & & & & \\ \vdots & & & (p_i^*, p_j^*)_1 & \\ 0 & & & & \end{bmatrix}$$

for c to be determined. Note that since $p_i\in m^\perp\cap n^\perp$ for $i\geqslant 2$, it follows from Lemma 6 that $(m^*, p_i^*)=(n^*, p_i^*)=0$. Furthermore, by construction, $\mathcal{G}$ and $\mathcal{H}$ agree on $m^\perp$.

We will choose c so that $\mathcal{H}$ and $\mathcal{G}$ agree on P.

Claim. There exists $\alpha\in\mathrm{GF}(l)$ such that $y=\langle m^*+\alpha n^*\rangle$ is a point of G and $\alpha\neq\alpha^\phi$.

There are $l-k$ values γ in GF (l) such that $\langle m^*+\gamma n^*\rangle$ is a point of $\mathcal{G}$, and only $k-1$ values $\gamma\neq 0$ such that $\gamma=\gamma^\phi$. Hence there exists such an α. Since $\langle y, z\rangle$ carries a $\mathcal{G}$ line of length $l-k$, we can choose a point t on $\langle y, z\rangle$ singular for $\mathcal{G}$, say $t=\langle m^*+\alpha n^*+\beta z^*\rangle$ for some $\beta\neq 0$. Now we solve for c. From the singularity of s and t, we have that $c+c^\phi=0$ and $\alpha c+(\alpha c)^\phi+\beta\beta^\phi(z^*, z^*)_1=0$. Thus, for $c=\beta\beta^\phi(z^*, z^*)/(\alpha^\phi-\alpha)$, we conclude from Corollary 10 that $\mathcal{G}$ and $\mathcal{H}$ agree on P.

We conclude the proof of this case by showing that if $x\in P_n\backslash m^\perp\cup P$, then x is singular for $\mathcal{G}$ exactly when $(x^*, x^*)=0$. By dimensionality the three dimensional subspace $\langle m, n, z, x\rangle$ meets $m^\perp\cap n^\perp$ in a line L through z. By Theorem 1, the agreement of $\mathcal{G}$ and $\mathcal{H}$ on P and L implies that $\mathcal{G}$ and $\mathcal{H}$ agree $\langle m, n, z, x\rangle$. Thus the singular points for $\mathcal{G}$ are exactly those for $\mathcal{H}$.

Case 2. *G contains an ultraaffine plane P' and no planes with semiovals.* Let P be the projective plane carrying P', and m, n be singular points for $\mathcal{G}$ in P such that $\langle m, n\rangle$ carries a $\mathcal{G}$ line of length $l-k$.

Claim. All the points in $m^{\perp}$ are singular for G. If z were a $\mathcal{G}$ point in $m^{\perp}$, then $\langle m, n, z\rangle$ would carry a plane with semioval, since $\langle m, n\rangle$ and $\langle m, z\rangle$ would carry $(l-k)$- and l-length lines, respectively.

We choose the basis $\{n^*, m^*=p_1^*, \dots, p_n^*\}$ as we did in case 1, and $c \neq 0 \in \mathrm{GF}(l)$ such that $c+c^{\Phi}=0$. The Hermitian form (,) defined on V_{n+1} with respect to this basis by

$$\begin{bmatrix} 0 & c & 0 & \cdots & 0 \\ c^{\Phi} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

produces a Hermitian geometry $\mathcal{H}$ agreeing with $\mathcal{G}$ on $m^{\perp}$ by construction and on $\langle m, n\rangle$ by Corollary 8. It follows that $\mathcal{G}$ and $\mathcal{H}$ agree on P.

To conclude that $\mathcal{G}$ and $\mathcal{H}$ agree on P_n, it is enough to consider any point $x \in P_n \backslash m^{\perp} \cup P$. The subspace $m^{\perp} \cap n^{\perp}$ meets $\langle m, n, x\rangle$ in a point z; thus $\langle m, z\rangle$ and $\langle n, z\rangle$ are deleted for both geometries. Furthermore, since $\mathcal{G}$ and $\mathcal{H}$ agree on $\langle m, n\rangle$, they agree on the ultraaffine plane carried by $\langle m, n, x\rangle$. Thus, x is singular for $\mathcal{G}$ exactly when x is singular for $\mathcal{H}$, i.e., $(x^*, x^*)=0$.

Case 3. *G contains all affine plans.* The results by Buekenhout [2] and Hall [7] imply that $\mathcal{G}$ is produced by the form defined using some basis of V_{n+1} by

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \blacksquare$$

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